OCU-PHYS-191
AP-GR-7

Energy-Momentum and Angular Momentum Carried by Gravitational Waves in Extended New General Relativity

Eisaku Sakane[1] and Toshiharu Kawai[2,3]

Department of Physics, Graduate School of Science, Osaka City University, Osaka 558-8585, Japan

Abstract

In an extended, new form of general relativity, which is a teleparallel theory of gravity, we examine the energy-momentum and angular momentum carried by gravitational wave radiated from Newtonian point masses in a weak-field approximation. The resulting wave form is identical to the corresponding wave form in general relativity, which is consistent with previous results in teleparallel theory. The expression for the dynamical energy-momentum density is identical to that for the canonical energy-momentum density in general relativity up to leading order terms on the boundary of a large sphere including the gravitational source, and the loss of dynamical energy-momentum, which is the generator of *internal* translations, is the same as that of the canonical energy-momentum in general relativity. Under certain asymptotic conditions for a non-dynamical Higgs-type field ψ^k, the loss of "spin" angular momentum, which is the generator of *internal* $SL(2,C)$ transformations, is the same as that of angular momentum in general relativity, and the losses of canonical energy-momentum and orbital angular momentum, which constitute the generator of Poincaré *coordinate* transformations, are vanishing.

[1]E-mail: `sakane@sci.osaka-cu.ac.jp`
[2]E-mail: `kawai@sci.osaka-cu.ac.jp`
[3]Present address: 6-5-5, Onodai, Osakasayama, Osaka 589-0023, Japan.

The results indicate that our definitions of the dynamical energy-momentum and angular momentum densities in this extended new general relativity work well for gravitational wave radiations, and the extended new general relativity accounts for the Hulse-Taylor measurement of the pulsar PSR1913+16.

1 Introduction

General relativity (GR) is a standard theory of gravity which has passed all the observational tests so far carried out, and it constitutes, together with quantum field theory, a basic framework of modern theoretical physics. In GR, however, it is usually asserted [1] that well-behaved energy-momentum and angular momentum densities cannot be defined in general for a gravitational field. For a restricted class of systems including asymptotically flat space-time, there exist tensor densities whose integrals over the cross section of the null infinity give the energy-momentum and angular momentum of the system in question [2].

There are many theories [3] that are potential alternatives to GR, including the $\overline{\text{Poincaré}}$ gauge theory ($\bar{\mathrm{P}}$GT) [4] and extended new general relativity (ENGR) [5]. $\bar{\mathrm{P}}$GT is formulated on the basis of the principal fiber bundle over the space-time possessing the covering group $\bar{P}_0$ of the proper orthochronous Poincaré group as the structure group, following the standard geometric formulation of Yang-Mills theories as closely as possible. ENGR is formulated as the teleparallel limit of $\bar{\mathrm{P}}$GT. The dynamical energy-momentum and "spin" angular momentum densities of gravitational and matter fields are all space-time vector densities, and their integrations over an arbitrary space-like surface σ are well-defined for any coordinate system employed [6].

For asymptotically flat space-time whose vierbeins satisfy certain asymptotic conditions,[4] the integration of the dynamical energy-momentum density over σ is the generator of *internal* translations and gives the total energy-momentum of the system. Also, the integration of the "spin" angular momentum density over σ is the generator of *internal* $SL(2,C)$-transformations and gives the *total* (=*spin*+*orbital*)

[4]With regard to the situation in ENGR, see Eqs. (2.45) and (2.46) in §2.2.

angular momentum in both theories. This holds in $\bar{\mathrm{P}}$GT when the Higgs-type field ψ^k satisfies the asymptotic condition $\psi^k = e^{(0)k}{}_\mu x^\mu + \psi^{(0)k} + O(1/r^\beta)$ with constants $e^{(0)k}{}_\mu, \psi^{(0)k}$ [7, 8, 9] and in ENGR when this asymptotic condition and certain other additional conditions are satisfied[5]. These theories describe within the uncertainties all the observed gravitational phenomena when the parameters in the gravitational Lagrangian densities satisfy certain conditions.

Direct observation of gravitational waves is one of the most challenging problems in present day gravitational physics. Several projects designed for this purpose are now being carried out, and gravitational radiations from various possible sources have been investigated theoretically, mainly on the basis of GR. However, also in classes of teleparallel theories of gravity, the form of gravitational waves is known to be identical to that of GR in post-Newtonian approximations [10, 11].

For the case in which a gravitational wave is radiated, however, the asymptotic behavior of vierbeins is different from that considered in Ref. [5] in general, and the question of whether our definitions of the energy-momentum and angular momentum densities work well in this case should also be answered.

The purpose of the present paper is to examine, in a weak-field approximation, the energy-momentum and angular momentum carried by gravitational waves radiated from Newtonian point masses. In §2, the basic framework of ENGR is briefly summarized as preparation for later discussion. In §3, the forms of the gravitational field equations and the dynamical energy-momentum density ${}^G\boldsymbol{T}_k{}^\mu$ of the gravitational field are given in the weak-field approximation. For plane wave solutions of the linearized homogeneous equations of a gravitational field, we give the average of ${}^G\boldsymbol{T}_k{}^\mu$ over a space-time region much larger than the inverse of the absolute value of the three-dimensional wave number vector. In §4, the quadrupole radiation formula for a gravitational wave emitted from a system of Newtonian point masses is obtained. In §5, we examine the emission rates of the dynamical energy-momentum and the angular momentum for two types of the asymptotic form of a Higgs-type field. Further, the emission rates of the canonical energy-momentum and the "extended orbital angular momentum" are examined. Finally, in §6, we give a summary and discussion.

2 Basic framework of extended new general relativity

2.1 $\overline{\text{Poincaré}}$ gauge theory

We first give the outline of $\bar{\text{P}}$GT, because ENGR is formulated as a reduction of this theory.

$\bar{\text{P}}$GT is formulated on the basis of the principal fiber bundle $\mathcal{P}$ over the space-time M possessing the covering group $\bar{P}_0$ of the proper orthochronous Poincaré group as the structure group. The space-time M is assumed to be a noncompact four-dimensional differentiable manifold with a countable base. The bundle $\mathcal{P}$ admits a connection Γ, the translational and rotational parts of whose coefficients will be written $A^k{}_\mu$ and $A^k{}_{l\mu}$, respectively. The fundamental field variables are $A^k{}_\mu$ and $A^k{}_{l\mu}$, the Higgs-type field is $\psi = \{\psi^k\}$, and the matter field is $\phi = \{\phi^A | A = 1, 2, \ldots, N\}$.[5] These fields transform according to[6]

$$\psi'^k = (\Lambda(a^{-1}))^k{}_m(\psi^m - t^m), \tag{2.1a}$$

$$A'^k{}_\mu = (\Lambda(a^{-1}))^k{}_m(A^m{}_\mu + t^m{}_{,\mu} + A^m{}_{n\mu}t^n), \tag{2.1b}$$

$$A'^k{}_{l\mu} = (\Lambda(a^{-1}))^k{}_m A^m{}_{n\mu}(\Lambda(a))^n{}_l + (\Lambda(a^{-1}))^k{}_m(\Lambda(a))^m{}_{l,\mu}\ , \tag{2.1c}$$

$$\phi'^A = [\rho((t,a)^{-1})]^A{}_B\phi^B\ , \tag{2.1d}$$

under the $\overline{\text{Poincaré}}$ gauge transformation

$$\begin{aligned} \sigma'(x) &= \sigma(x) \cdot [t(x), a(x)]\ , \\ t(x) &\in T^4, \quad a(x) \in SL(2, C). \end{aligned} \tag{2.2}$$

Here, Λ is the covering map from $SL(2, C)$ to the proper orthochronous Lorentz group, and ρ denotes the representation of the $\overline{\text{Poincaré}}$ group to which the field ϕ^A

[5]Unless otherwise stated, we use the following conventions for indices. Letters from the middle part of the Greek alphabet, λ, μ, ν, $\ldots$, and from the middle part of the Latin alphabet, k, l, m, $\ldots$, take the values 0, 1, 2 and 3. The capital letters A and B are used as indices for components of the field ϕ, and N denotes the dimension of the representation ρ.

[6]For the function f on M, we define $f_{,\mu} \overset{\text{def}}{=} \partial f / \partial x^\mu$.

belongs. Also, σ and σ' stand for local cross sections of $\mathcal{P}$. The dual components $e^k{}_\mu$ of the vierbein fields $e^\mu{}_k\partial/\partial x^\mu$ are related to the field ψ^k and the gauge potentials $A^k{}_\mu$ and $A^k{}_{l\mu}$ through the relation

$$e^k{}_\mu = \psi^k{}_{,\mu} + A^k{}_{l\mu}\psi^l + A^k{}_\mu \;, \tag{2.3}$$

and these transform according to

$$e'^k{}_\mu = (\Lambda(a^{-1}))^k{}_l e^l{}_\mu \;, \tag{2.4}$$

under the transformation (2.2). Also, they are related to the metric $g_{\mu\nu}dx^\mu \otimes dx^\nu$ of M through the relation

$$g_{\mu\nu} = e^k{}_\mu \eta_{kl} e^l{}_\nu \;, \tag{2.5}$$

with $(\eta_{kl}) \stackrel{\text{def}}{=} \text{diag}(-1,1,1,1)$.

The field strengths $R^k{}_{l\mu\nu}, R^k{}_{\mu\nu}$ and $T^k{}_{\mu\nu}$ of $A^k{}_{l\mu}, A^k{}_\mu$ and $e^k{}_\mu$ are given by[7]

$$R^k{}_{l\mu\nu} \stackrel{\text{def}}{=} 2(A^k{}_{l[\nu,\mu]} + A^k{}_{m[\mu}A^m{}_{l\nu]}), \tag{2.6a}$$

$$R^k{}_{\mu\nu} \stackrel{\text{def}}{=} 2(A^k{}_{[\nu,\mu]} + A^k{}_{l[\mu}A^l{}_{\nu]}), \tag{2.6b}$$

$$T^k{}_{\mu\nu} \stackrel{\text{def}}{=} 2(e^k{}_{[\nu,\mu]} + A^k{}_{l[\mu}e^l{}_{\nu]}), \tag{2.6c}$$

respectively, and we have the relation

$$T^k{}_{\mu\nu} = R^k{}_{\mu\nu} + R^k{}_{l\mu\nu}\psi^l \;. \tag{2.7}$$

The field strengths $T^k{}_{\mu\nu}$ and $R^k{}_{l\mu\nu}$ are both invariant under *internal* translations.

There is a 2 to 1 bundle homomorphism F from $\mathcal{P}$ to the affine frame bundle $\mathcal{A}(M)$ over M, and there exist an extended spinor structure and a spinor structure associated with it [12]. The space-time M is orientable, which follows from its assumed noncompactness and the fact that M has a spinor structure.

[7] We define

$$A_{\ldots[\mu\ldots\nu]\ldots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\ldots\mu\ldots\nu\ldots} - A_{\ldots\nu\ldots\mu\ldots}),$$

$$A_{\ldots(\mu\ldots\nu)\ldots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\ldots\mu\ldots\nu\ldots} + A_{\ldots\nu\ldots\mu\ldots}).$$

The affine frame bundle $\mathcal{A}(M)$ admits a connection Γ_A. The T^4 part $\Gamma^\mu{}_\nu$ and the $GL(4,R)$ part $\Gamma^\lambda{}_{\mu\nu}$ of its connection coefficients are related to $A^k{}_{l\mu}$ and $e^k{}_\mu$ through the relations

$$\Gamma^\mu{}_\nu = \delta^\mu{}_\nu \ , \tag{2.8a}$$

$$A^k{}_{l\mu} = e^k{}_\lambda e^\nu{}_l \Gamma^\lambda{}_{\nu\mu} + e^k{}_\nu e^\nu{}_{l,\mu} \ , \tag{2.8b}$$

by the requirement that F maps the connection Γ into Γ_A, and the space-time M is of the Riemann-Cartan type.

The torsion is given by

$$T^\lambda{}_{\mu\nu} \stackrel{\text{def}}{=} 2\Gamma^\lambda{}_{[\nu\mu]} \ , \tag{2.9}$$

and the T^4 and $GL(4,R)$ parts of the curvature are given by

$$R^\lambda{}_{\mu\nu} = 2(\Gamma^\lambda{}_{[\nu,\mu]} + \Gamma^\lambda{}_{\rho[\mu}\Gamma^\rho{}_{\nu]}), \tag{2.10}$$

$$R^\lambda{}_{\rho\mu\nu} = 2(\Gamma^\lambda{}_{\rho[\nu,\mu]} + \Gamma^\lambda{}_{\tau[\mu}\Gamma^\tau{}_{\rho\nu]}), \tag{2.11}$$

respectively. Then, we have the relations

$$T^k{}_{\mu\nu} = e^k{}_\lambda T^\lambda{}_{\mu\nu} = e^k{}_\lambda R^\lambda{}_{\mu\nu} \ , \tag{2.12}$$

$$R^k{}_{l\mu\nu} = e^k{}_\lambda e^\rho{}_l R^\lambda{}_{\rho\mu\nu} \ , \tag{2.13}$$

which follow from Eq. (2.8).

The covariant derivative of the matter field ϕ takes the form

$$D_k\phi^A = e^\mu{}_k D_\mu \phi^A \ , \tag{2.14}$$

$$D_\mu\phi^A \stackrel{\text{def}}{=} \partial_\mu\phi^A + \frac{i}{2}A^{kl}{}_\mu (M_{kl}\phi)^A + iA^k{}_\mu (P_k\phi)^A \ , \tag{2.15}$$

where M_{kl} and P_k are representation matrices of the standard basis of the Lie algebra of the group $\bar{P}_0$: $M_{kl} = -i\rho_*(\bar{M}_{kl}), P_k = -i\rho_*(\bar{P}_k)$. The matrix P_k represents the "intrinsic energy-momentum" of the field ϕ^A [12], and it is vanishing for all observed fields.

2.2 Extended new general relativity

In $\bar{\mathrm{P}}$GT, we consider the case in which the field strength $R^{kl}{}_{\mu\nu}$ vanishes identically,

$$R^{kl}{}_{\mu\nu} \equiv 0. \tag{2.16}$$

Thus, the curvature $R^{\lambda}{}_{\rho\mu\nu}$ vanishes, and we have a teleparallel theory.

By choosing the $SL(2,C)$-gauge such that

$$A^{kl}{}_{\mu} \equiv 0, \tag{2.17}$$

the following reduced expressions are obtained:

$$e^{k}{}_{\mu} = \psi^{k}{}_{,\mu} + A^{k}{}_{\mu} \ , \tag{2.18}$$

$$\Gamma^{\lambda}{}_{\mu\nu} = e^{\lambda}{}_{k} e^{k}{}_{\mu,\nu} \ , \tag{2.19}$$

$$D_k \phi^A = e^{\mu}{}_{k} D_\mu \phi^A \ , \tag{2.20}$$

$$D_\mu \phi^A = \partial_\mu \phi^A + i A^{k}{}_{\mu} (P_k \phi)^A \ . \tag{2.21}$$

The Lagrangian takes the form[8]

$$L = L^T(T_{klm}) + L^M(e^{k}{}_{\mu}, \psi^k, D_k\phi^A, \phi^A), \tag{2.22}$$

where L^M is the Lagrangian of the matter field ϕ^A and L^T is the gravitational Lagrangian. We impose the following requirements: (R.i) L is invariant under the transformation (2.2) with arbitrary functions t^k and an arbitrary *constant* element a of $SL(2,C)$; (R.ii) The functional L is a scalar field on M.

The gravitational Lagrangian [13]

$$L^T \stackrel{\text{def}}{=} c_1 t^{klm} t_{klm} + c_2 v^k v_k + c_3 a^k a_k \tag{2.23}$$

satisfies these requirements, where c_1, c_2 and c_3 are real constants. The quantities t_{klm}, v_k and a_k are the irreducible components of T_{klm} defined by

$$t_{klm} \stackrel{\text{def}}{=} \frac{1}{2}(T_{klm} + T_{lkm}) + \frac{1}{6}(\eta_{mk} v_l + \eta_{ml} v_k) - \frac{1}{3}\eta_{kl} v_m \ , \tag{2.24}$$

$$v_k \stackrel{\text{def}}{=} T^{l}{}_{lk} \ , \tag{2.25}$$

$$a_k \stackrel{\text{def}}{=} \frac{1}{6}\epsilon_{klmn} T^{lmn} \ , \tag{2.26}$$

[8]The field components $e^{k}{}_{\mu}$ and $e^{\mu}{}_{k}$ are used to convert Latin and Greek indices. Also, the raising and lowering of the indices k, l, m, $\ldots$ are accomplished through use of $(\eta^{kl}) = (\eta_{kl})^{-1}$ and (η_{kl}).

where the symbol ϵ_{klmn} represents the Levi-Civita tensor, with $\epsilon_{(0)(1)(2)(3)} = -1$.[9]

If the parameters c_1, c_2 and c_3 satisfy

$$c_1 = -c_2 = \frac{4}{9}c_3 = -\frac{1}{3\kappa} \ , \tag{2.27}$$

where κ is the Einstein gravitational constant, $\kappa \stackrel{\text{def}}{=} 8\pi G/c^4$,[10] the gravitational part of the action integral is equal to the Einstein-Hilbert action integral, namely [13]

$$\int d^4x\sqrt{-g}L^T = \int d^4x\frac{1}{2\kappa}\sqrt{-g}R(\{\}), \tag{2.28}$$

where $R(\{\})$ denotes the Riemann-Christoffel scalar curvature. Here, we should mention that even in the case that the condition (2.27) is satisfied, our theory does not reduce to GR, because the couplings of matter fields (the spinor field, for example) with the gravitational field are different from those in GR.

In defining the energy-momentum and angular momentum, there are two possibilities for choosing the set of independent field variables [5, 7, 8, 9]: One is to choose the set $\{\psi^k, A^k{}_\mu, \phi^A\}$, and the other is to choose the set $\{\psi^k, e^k{}_\mu, \phi^A\}$. In the rest of this paper, we employ $\{\psi^k, A^k{}_\mu, \phi^A\}$ as the set of independent field variables, because this choice is superior to the other, as shown in Refs. [5, 7, 8, 9].

From the requirement (R.i), we obtain the identities[11]

$$\frac{\delta \boldsymbol{L}}{\delta\psi^k} + \partial_\mu\left(\frac{\delta \boldsymbol{L}}{\delta A^k{}_\mu}\right) + i\frac{\delta \boldsymbol{L}}{\delta\phi^A}(P_k\phi)^A \equiv 0, \tag{2.29}$$

$$\boldsymbol{F}_k{}^{(\mu\nu)} \equiv 0, \tag{2.30}$$

$${}^{\text{tot}}\boldsymbol{T}_k{}^\mu - \partial_\nu \boldsymbol{F}_k{}^{\mu\nu} - \frac{\delta \boldsymbol{L}}{\delta A^k{}_\mu} \equiv 0, \tag{2.31}$$

$$\partial_\mu{}^{\text{tot}}\boldsymbol{S}_{kl}{}^\mu - 2\frac{\delta \boldsymbol{L}}{\delta\psi^{[k}}\psi_{l]} - 2\frac{\delta \boldsymbol{L}}{\delta A^{[k}{}_\mu}A_{l]\mu} - i\frac{\delta \boldsymbol{L}}{\delta\phi^A}(M_{kl}\phi)^A \equiv 0, \tag{2.32}$$

[9] Latin indices are put in parentheses to distinguish them from Greek indices.

[10] G and c stand for the Newtonian gravitational constant and the light velocity in vacuum, respectively.

[11] For instance, $\delta\boldsymbol{L}/\delta\psi^k$ denotes the Euler derivative with respect to ψ^k.

where we have defined

$$\boldsymbol{L} \stackrel{\text{def}}{=} \sqrt{-g}L \ , \quad g \stackrel{\text{def}}{=} \det(g_{\mu\nu}) \ , \tag{2.33}$$

$$\boldsymbol{F}_k{}^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{L}}{\partial A^k{}_{\mu,\nu}} \ , \tag{2.34}$$

$$^{\text{tot}}\boldsymbol{T}_k{}^{\mu} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{L}}{\partial \psi^k{}_{,\mu}} + i\frac{\partial \boldsymbol{L}}{\partial \phi^A{}_{,\mu}}(P_k\phi)^A \ , \tag{2.35}$$

$$^{\text{tot}}\boldsymbol{S}_{kl}{}^{\mu} \stackrel{\text{def}}{=} -2\frac{\partial \boldsymbol{L}}{\partial \psi^{[k}{}_{,\mu}}\psi_{l]} - 2\boldsymbol{F}_{[k}{}^{\nu\mu}A_{l]\nu} - i\frac{\partial \boldsymbol{L}}{\partial \phi^A{}_{,\mu}}(M_{kl}\phi)^A \ . \tag{2.36}$$

If the field equations $\delta\boldsymbol{L}/\delta A^k{}_\mu = 0$ and $\delta\boldsymbol{L}/\delta\phi^A = 0$ are both satisfied, we have the following:

- The field equation $\delta\boldsymbol{L}/\delta\psi^k = 0$ is automatically satisfied, and thus ψ^k is not an independent dynamical field variable.

- There are two conservation laws,

$$\partial_\mu{}^{\text{tot}}\boldsymbol{T}_k{}^{\mu} = 0, \tag{2.37}$$

$$\partial_\mu{}^{\text{tot}}\boldsymbol{S}_{kl}{}^{\mu} = 0, \tag{2.38}$$

  which follow from Eqs. (2.30)–(2.32).

  The former is the differential conservation law of the dynamical energy-momentum, and the latter is that of the "spin" angular momentum.

We split the densities $^{\text{tot}}\boldsymbol{T}_k{}^{\mu}$ and $^{\text{tot}}\boldsymbol{S}_{kl}{}^{\mu}$ into gravitational and matter parts as

$$^{\text{tot}}\boldsymbol{T}_k{}^{\mu} = {}^G\boldsymbol{T}_k{}^{\mu} + {}^M\boldsymbol{T}_k{}^{\mu} \ , \tag{2.39}$$

$$^{\text{tot}}\boldsymbol{S}_{kl}{}^{\mu} = {}^G\boldsymbol{S}_{kl}{}^{\mu} + {}^M\boldsymbol{S}_{kl}{}^{\mu} \ , \tag{2.40}$$

where we have defined

$$^G\boldsymbol{T}_k{}^{\mu} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{L}^T}{\partial \psi^k{}_{,\mu}} = \frac{\partial \boldsymbol{L}^T}{\partial A^k{}_\mu} \ , \tag{2.41}$$

$$^M\boldsymbol{T}_k{}^{\mu} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{L}^M}{\partial \psi^k{}_{,\mu}} + i\frac{\partial \boldsymbol{L}^M}{\partial \phi^A{}_{,\mu}}(P_k\phi)^A = \frac{\partial \boldsymbol{L}^M}{\partial A^k{}_\mu} \ , \tag{2.42}$$

$$^G\boldsymbol{S}_{kl}{}^{\mu} \stackrel{\text{def}}{=} -2\frac{\partial \boldsymbol{L}^T}{\partial \psi^{[k}{}_{,\mu}}\psi_{l]} - 2\boldsymbol{F}_{[k}{}^{\nu\mu}A_{l]\nu} \ , \tag{2.43}$$

$$^M\boldsymbol{S}_{kl}{}^{\mu} \stackrel{\text{def}}{=} -2\frac{\partial \boldsymbol{L}^M}{\partial \psi^{[k}{}_{,\mu}}\psi_{l]} - i\frac{\partial \boldsymbol{L}^M}{\partial \phi^A{}_{,\mu}}(M_{kl}\phi)^A \ , \tag{2.44}$$

with $\boldsymbol{L}^T \stackrel{\text{def}}{=} \sqrt{-g}L^T$ and $\boldsymbol{L}^M \stackrel{\text{def}}{=} \sqrt{-g}L^M$. Here, ${}^G\boldsymbol{T}_k{}^\mu$ and ${}^M\boldsymbol{T}_k{}^\mu$ are the dynamical energy-momentum densities of the gravitational field and the matter field, respectively, while ${}^G\boldsymbol{S}_{kl}{}^\mu$ and ${}^M\boldsymbol{S}_{kl}{}^\mu$ are the "spin" angular momentum densities of the gravitational field and the matter field, respectively. The densities ${}^G\boldsymbol{T}_k{}^\mu$, ${}^M\boldsymbol{T}_k{}^\mu$, ${}^G\boldsymbol{S}_{kl}{}^\mu$ and ${}^M\boldsymbol{S}_{kl}{}^\mu$ are all space-time vector densities [6].

In Ref. [5], the integrals of the dynamical energy-momentum and "spin" angular momentum densities over a space-like surface σ are examined for vierbeins with the asymptotic behavior described below.

⟨1⟩ The components $e^k{}_\mu$ of the vierbein fields possess the asymptotic property[12]

$$e^k{}_\mu = e^{(0)k}{}_\mu + f^k{}_\mu \ , \quad f^k{}_{\mu,(m)} = O(1/r^{1+m}) \quad (m = 0, 1, 2), \tag{2.45}$$

where $f^k{}_{\mu,(m)}$ denotes the mth order partial derivative with respect to x^λ, and the $e^{(0)k}{}_\mu$ are constant vierbeins satisfying $e^{(0)k}{}_\mu \eta_{kl} e^{(0)l}{}_\nu = \eta_{\mu\nu}$.

⟨2⟩ The antisymmetric part of components $f_{\mu\nu} \stackrel{\text{def}}{=} e^{(0)k}{}_\mu \eta_{kl} f^l{}_\nu$ satisfy

$$f_{[\mu\nu],(m)} = O(1/r^{1+\alpha+m}) \quad (m = 0, 1), \tag{2.46}$$

where α is positive but otherwise arbitrary.

It has been shown that

$$M_k \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\boldsymbol{T}_k{}^\mu d\sigma_\mu = e^{(0)\mu}{}_k M_\mu \ , \tag{2.47}$$

$$S_{kl} \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\boldsymbol{S}_{kl}{}^\mu d\sigma_\mu = e^{(0)}{}_{k\mu} e^{(0)}{}_{l\nu} M^{\mu\nu} + 2\psi^{(0)}{}_{[k} M_{l]} \ , \tag{2.48}$$

where $d\sigma_\mu$ denotes the surface element on σ. In the above, we have defined

$$M_\mu \stackrel{\text{def}}{=} \eta_{\mu\nu} \int_\sigma \theta^{\nu\lambda} d\sigma_\lambda \tag{2.49}$$

$$M^{\mu\nu} \stackrel{\text{def}}{=} \int_\sigma \partial_\rho K^{\mu\nu\lambda\rho} d\sigma_\lambda = \int_\sigma (x^\mu \theta^{\nu\lambda} - x^\nu \theta^{\mu\lambda}) d\sigma_\lambda \ , \tag{2.50}$$

[12]The expression $O(1/r^n)$ with real n denotes a term for which $r^n O(1/r^n)$ remains finite for $r \to \infty$; a term denoted as $O(1/r^n)$ could, of course, also be zero.

with

$$\theta^{\nu\lambda} \stackrel{\text{def}}{=} \frac{1}{\kappa}\partial_\rho\partial_\sigma\{(-g)g^{\nu[\lambda}g^{\rho]\sigma}\}, \tag{2.51}$$

$$K^{\mu\nu\lambda\rho} \stackrel{\text{def}}{=} \frac{1}{\kappa}\left(x^\mu\partial_\sigma\{(-g)g^{\nu[\lambda}g^{\rho]\sigma}\} - x^\nu\partial_\sigma\{(-g)g^{\mu[\lambda}g^{\rho]\sigma}\} + (-g)g^{\mu[\lambda}g^{\rho]\nu} - \eta^{\mu[\lambda}\eta^{\rho]\nu}\right). \tag{2.52}$$

This expression of $\theta^{\nu\lambda}$ is the same as that of the symmetric energy-momentum density proposed by Landau and Lifshitz [14]. Equation (2.48) has been obtained by choosing the asymptotic form of the Higgs-type field ψ^k as

$$\psi^k = e^{(0)k}{}_\mu x^\mu + \psi^{(0)k} + O\left(\frac{1}{r^\beta}\right), \tag{2.53a}$$

$$\psi^k{}_{,\mu} = e^{(0)k}{}_\mu + O\left(\frac{1}{r^{1+\beta}}\right), \quad (\beta > 0) \tag{2.53b}$$

$$\psi^k{}_{,\mu\nu} = O\left(\frac{1}{r^2}\right), \tag{2.53c}$$

with $\psi^{(0)k}$ and β constant, whereas Eq. (2.47) has been obtained without imposing the conditions (2.53).

From the requirement (R.ii), we obtain the identity

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu - \partial_\lambda \boldsymbol{\Psi}_\mu{}^{\nu\lambda} - \frac{\delta \boldsymbol{L}}{\delta A^k{}_\nu} A^k{}_\mu \equiv 0, \tag{2.54}$$

with

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \boldsymbol{L} - \boldsymbol{F}_k{}^{\lambda\nu} A^k{}_{\lambda,\mu} - \frac{\partial \boldsymbol{L}}{\partial \phi^A{}_{,\nu}}\phi^A{}_{,\mu} - \frac{\partial \boldsymbol{L}}{\partial \psi^k{}_{,\nu}}\psi^k{}_{,\mu}\,, \tag{2.55}$$

$$\boldsymbol{\Psi}_\mu{}^{\nu\lambda} \stackrel{\text{def}}{=} \boldsymbol{F}_k{}^{\nu\lambda} A^k{}_\mu = -\boldsymbol{\Psi}_\mu{}^{\lambda\nu}\,. \tag{2.56}$$

The identity (2.54) leads to

$$\partial_\nu \widetilde{\boldsymbol{T}}_\mu{}^\nu = 0, \tag{2.57}$$

$$\partial_\lambda \widetilde{\boldsymbol{M}}_\mu{}^{\nu\lambda} = 0 \tag{2.58}$$

when $\delta\boldsymbol{L}/\delta A^k{}_\nu = 0$, where we have defined

$$\widetilde{\boldsymbol{M}}_\mu{}^{\nu\lambda} \stackrel{\text{def}}{=} 2(\boldsymbol{\Psi}_\mu{}^{\nu\lambda} - x^\nu \widetilde{\boldsymbol{T}}_\mu{}^\lambda). \tag{2.59}$$

Equations (2.57) and (2.58) are the differential conservation laws of the canonical energy-momentum and the "extended orbital angular momentum" defined by

$$M^c{}_\mu \stackrel{\text{def}}{=} \int_\sigma \widetilde{\boldsymbol{T}}_\mu{}^\nu d\sigma_\nu \ , \quad L_\mu{}^\nu \stackrel{\text{def}}{=} \int_\sigma \widetilde{\boldsymbol{M}}_\mu{}^{\nu\lambda} d\sigma_\lambda \ , \tag{2.60}$$

respectively [5]. The canonical energy-momentum and the "extended orbital angular momentum" are the generators of general affine *coordinate* transformations. The antisymmetric part $L_{[\mu\nu]} \stackrel{\text{def}}{=} L_{[\mu}{}^\lambda \eta_{\lambda\nu]}$ is the orbital angular momentum[13] and is the generator of *coordinate* Lorentz transformation [5].

We split the canonical energy-momentum density into gravitational and matter parts as

$$\widetilde{\boldsymbol{T}}_\mu{}^\nu = {}^G\widetilde{\boldsymbol{T}}_\mu{}^\nu + {}^M\boldsymbol{T}_\mu{}^\nu \ , \tag{2.61}$$

where we have defined

$${}^G\widetilde{\boldsymbol{T}}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \boldsymbol{L}^T - \boldsymbol{F}_k{}^{\lambda\nu} A^k{}_{\lambda,\mu} - \frac{\partial \boldsymbol{L}^T}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu} \ , \tag{2.62}$$

$${}^M\boldsymbol{T}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \boldsymbol{L}^M - \frac{\partial \boldsymbol{L}^M}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu} - \frac{\partial \boldsymbol{L}^M}{\partial \phi^A{}_{,\nu}} \phi^A{}_{,\mu} \ . \tag{2.63}$$

The density ${}^G\widetilde{\boldsymbol{T}}_\mu{}^\nu$ does not transform as a tensor density under general coordinate transformations, while ${}^M\boldsymbol{T}_\mu{}^\nu$ does transform as a tensor density.[6]

As is described in Ref. [5], the generators $M^c{}_\mu$ and $L_\mu{}^\nu$ vanish for vierbeins with the asymptotic forms satisfying Eqs. (2.45) and (2.46) when the condition (2.53) is satisfied.

The field equation $\delta\boldsymbol{L}/\delta A^k{}_\mu = 0$ has the expression

$$-2\nabla_\lambda F^{\mu\nu\lambda} + 2v_\lambda F^{\mu\nu\lambda} + 2H^{\mu\nu} - g^{\mu\nu} L^T = T^{\mu\nu} \ , \tag{2.64}$$

where we have defined

$$\nabla_\lambda F^{\mu\nu\lambda} \stackrel{\text{def}}{=} \partial_\lambda F^{\mu\nu\lambda} + \Gamma^\mu{}_{\sigma\lambda} F^{\sigma\nu\lambda} + \Gamma^\nu{}_{\sigma\lambda} F^{\mu\sigma\lambda} + \Gamma^\lambda{}_{\sigma\lambda} F^{\mu\nu\sigma} \ , \tag{2.65}$$

$$F^{\mu\nu\lambda} \stackrel{\text{def}}{=} c_1(t^{\mu\nu\lambda} - t^{\mu\lambda\nu}) + c_2(g^{\mu\nu} v^\lambda - g^{\mu\lambda} v^\nu) - \frac{1}{3} c_3 \epsilon^{\mu\nu\lambda\rho} a_\rho \ , \tag{2.66}$$

$$H^{\mu\nu} \stackrel{\text{def}}{=} T^{\rho\sigma\mu} F_{\rho\sigma}{}^\nu - \frac{1}{2} T^{\nu\rho\sigma} F^\mu{}_{\rho\sigma} \ . \tag{2.67}$$

[13] This $L_{[\mu\nu]}$ should not be confused with the orbital part of the "spin" angular momentum S_{kl}.

Also, $T^{\mu\nu}$ is the energy-momentum density of the gravitational source defined by

$$\sqrt{-g}T^{\mu\nu} \stackrel{\text{def}}{=} \eta^{kl} e^{\mu}{}_{l} \frac{\delta \boldsymbol{L}^M}{\delta A^k{}_{\nu}} \ . \tag{2.68}$$

3 Weak-field approximation

We now consider weak field situations in which the vierbein fields $e^k{}_\mu$ take the form

$$e^k{}_\mu = e^{(0)k}{}_\mu + f^k{}_\mu \ , \quad |f^k{}_\mu| \ll 1, \tag{3.1}$$

where the $e^{(0)k}{}_\mu$ are constant vierbeins satisfying $e^{(0)k}{}_\mu \eta_{kl} e^{(0)l}{}_\nu = \eta_{\mu\nu}$. The components of the metric and torsion tensors are given by, up to terms linear in $f^k{}_\mu$,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2 f_{(\mu\nu)} \ , \tag{3.2}$$

$$T_{\lambda\mu\nu} = \partial_\mu f_{\lambda\nu} - \partial_\nu f_{\lambda\mu} \ , \tag{3.3}$$

where we have defined $f_{\mu\nu} \stackrel{\text{def}}{=} e^{(0)k}{}_\mu \eta_{kl} f^l{}_\nu$.[14]

In the weak-field approximation, the symmetric and antisymmetric parts of the field equation (2.64) take the form

$$\begin{aligned} 3c_1 & \left\{ \Box \bar{f}_{(\mu\nu)} - \partial^\lambda (\partial_\mu \bar{f}_{(\nu\lambda)} + \partial_\nu \bar{f}_{(\mu\lambda)}) + \eta_{\mu\nu} \partial_\rho \partial_\sigma \bar{f}^{(\rho\sigma)} \right\} \\ & + (c_1 + c_2) \left\{ -\eta_{\mu\nu} \Box \bar{f} - 2\eta_{\mu\nu} \partial_\rho \partial_\sigma \bar{f}^{(\rho\sigma)} + \partial_\mu \partial_\nu \bar{f} \right. \\ & \left. + \partial^\lambda (\partial_\mu \bar{f}_{(\nu\lambda)} + \partial_\nu \bar{f}_{(\mu\lambda)}) - \partial^\lambda (\partial_\mu f_{[\nu\lambda]} + \partial_\nu f_{[\mu\lambda]}) \right\} = T_{(\mu\nu)} \ , \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \left(c_1 - \frac{4}{9} c_3 \right) \left\{ \Box f_{[\mu\nu]} + \partial^\lambda (\partial_\mu f_{[\nu\lambda]} - \partial_\nu f_{[\mu\lambda]}) \right\} \\ & + (c_1 + c_2) \left\{ \partial^\lambda (\partial_\mu \bar{f}_{(\nu\lambda)} - \partial_\nu \bar{f}_{(\mu\lambda)}) - \partial^\lambda (\partial_\mu f_{[\nu\lambda]} - \partial_\nu f_{[\mu\lambda]}) \right\} = T_{[\mu\nu]} \ , \end{aligned} \tag{3.5}$$

with $\Box \stackrel{\text{def}}{=} \partial^\mu \partial_\mu$. Here, we have introduced

$$\bar{f}_{(\mu\nu)} \stackrel{\text{def}}{=} f_{(\mu\nu)} - \frac{1}{2} \eta_{\mu\nu} f \ , \quad f \stackrel{\text{def}}{=} \eta^{\mu\nu} f_{(\mu\nu)} \ , \tag{3.6}$$

$$\bar{f} \stackrel{\text{def}}{=} \eta^{\mu\nu} \bar{f}_{(\mu\nu)} \ . \tag{3.7}$$

[14]For $f^k{}_\mu$, we use the convention that both the Greek and Latin indices of $f^k{}_\mu$ are raised or lowered with the Minkowski metric, and that they are converted into one another with $e^{(0)k}{}_\mu$ or $e^{(0)\mu}{}_k$, where $(e^{(0)\mu}{}_k) \stackrel{\text{def}}{=} (e^{(0)k}{}_\mu)^{-1}$. Thus, $f^{\mu\nu}$, for example, represents $\eta^{\nu\lambda} e^{(0)\mu}{}_k f^k{}_\lambda (\neq g^{\nu\lambda} e^\mu{}_k f^k{}_\lambda)$.

We consider the energy-momentum density of the source $T_{\mu\nu}$ to lowest order in $f^k{}_\mu$. Therefore it is independent of $f^k{}_\mu$ and satisfies the ordinary conservation law in special relativity,

$$\partial^\nu T_{\mu\nu} = 0. \tag{3.8}$$

Let us consider the transformations

$$f'_{(\mu\nu)} = f_{(\mu\nu)} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu \ , \tag{3.9}$$

$$f'_{[\mu\nu]} = f_{[\mu\nu]} + \partial_\mu \chi_\nu - \partial_\nu \chi_\mu \ , \tag{3.10}$$

where ε_μ and χ_μ are arbitrary small functions. Since Eqs. (3.4) and (3.5) are invariant under the transformations (3.9) and (3.10) with $\varepsilon_\mu = \chi_\mu$, we can impose the harmonic coordinate condition

$$\partial_\nu \bar{f}^{(\mu\nu)} = 0. \tag{3.11}$$

The Lagrangian L^T with the parameters c_1 and c_2 satisfying

$$c_1 = -c_2 = -\frac{1}{3\kappa} \tag{3.12}$$

compares quite favorably with experiment [13]. We therefore assume (3.12) to hold henceforth. Under the conditions (3.11) and (3.12), Eq. (3.5) reduces to

$$\left(c_1 - \frac{4}{9}c_3\right)\left\{\Box f_{[\mu\nu]} + \partial^\lambda(\partial_\mu f_{[\nu\lambda]} - \partial_\nu f_{[\mu\lambda]})\right\} = T_{[\mu\nu]} \ , \tag{3.13}$$

which is still invariant under the transformation (3.10) [13]. Thus, we can impose the condition

$$\partial_\nu f^{[\mu\nu]} = 0. \tag{3.14}$$

Finally, under the conditions (3.11), (3.12) and (3.14), the field equations of $\bar{f}_{(\mu\nu)}$ and $f_{[\mu\nu]}$ become

$$\Box \bar{f}_{(\mu\nu)} = -\kappa T_{(\mu\nu)} \ , \tag{3.15}$$

$$\Box f_{[\mu\nu]} = -\lambda T_{[\mu\nu]} \ , \tag{3.16}$$

where we have defined $1/\lambda \stackrel{\text{def}}{=} -c_1 + (4/9)c_3 \neq 0$. From Eqs. (3.11) and (3.15), we find that the symmetric part of $T_{\mu\nu}$ satisfies the conservation law

$$\partial^\nu T_{(\mu\nu)} = 0, \tag{3.17}$$

and the antisymmetric part of $T_{\mu\nu}$ satisfies

$$\partial^\nu T_{[\mu\nu]} = 0, \tag{3.18}$$

which follows from Eqs. (3.14) and (3.16) [13].

Let us consider the plane wave solutions of Eqs. (3.15) and (3.16) with $T_{\mu\nu} \equiv 0$,

$$\bar{f}_{(\mu\nu)}(\boldsymbol{x}, x^0) = \mathcal{U}_{\mu\nu} e^{ik\cdot x} + \bar{\mathcal{U}}_{\mu\nu} e^{-ik\cdot x}\ , \tag{3.19}$$

$$f_{[\mu\nu]}(\boldsymbol{x}, x^0) = \mathcal{V}_{\mu\nu} e^{ik\cdot x} + \bar{\mathcal{V}}_{\mu\nu} e^{-ik\cdot x}\ , \tag{3.20}$$

where $k \cdot x \stackrel{\text{def}}{=} k_\mu x^\mu$. Here, $\mathcal{U}_{\mu\nu}$ and $\mathcal{V}_{\mu\nu}$ are constant amplitudes, $\bar{\mathcal{U}}$ and $\bar{\mathcal{V}}$ are their complex conjugates, and k_μ is a constant wave vector, which satisfy the relations

$$k_\mu k^\mu = 0, \quad \mathcal{U}_{\mu\nu} k^\nu = 0, \quad \mathcal{V}_{\mu\nu} k^\nu = 0. \tag{3.21}$$

Following the prescription given in Section 35.4 of Ref. [1], we impose the transverse-traceless gauge condition

$$\mathcal{U}_{\mu\nu}\zeta^\nu = 0, \quad \mathcal{U}^\mu{}_\mu = 0, \quad \mathcal{V}_{\mu\nu}\zeta^\nu = 0, \tag{3.22}$$

where ζ^μ is a constant time-like vector. We see that the number of physically significant components of $\bar{f}_{(\mu\nu)}$ is two, while that of $f_{[\mu\nu]}$ is one.

We next calculate the energy-momentum of the plane waves given by Eqs. (3.19) and (3.20). The dynamical energy-momentum density ${}^G\boldsymbol{T}_l{}^\mu$ of gravitational field

has, to lowest order in $f_{\mu\nu}$, the expression

$$
\begin{aligned}
2\kappa\, {}^G\boldsymbol{T}_l{}^{\mu} = e^{(0)\mu}{}_l \Big[& -\partial^\sigma \bar f^{(\lambda\rho)}\partial_\sigma \bar f_{(\lambda\rho)} + \partial^\sigma \bar f^{(\lambda\rho)}\partial_\rho \bar f_{(\lambda\sigma)} + \frac{1}{2}\partial^\sigma \bar f \partial_\sigma \bar f + 2\partial^\sigma \bar f^{(\lambda\rho)}\partial_\rho f_{[\lambda\sigma]} \\
& + \partial^\lambda f^{[\sigma\rho]}\partial_\rho f_{[\lambda\sigma]} - \frac{\kappa}{\lambda}\Big(\partial^\sigma f^{[\lambda\rho]}\partial_\sigma f_{[\lambda\rho]} + 2\partial^\lambda f^{[\sigma\rho]}\partial_\rho f_{[\lambda\sigma]}\Big)\Big] \\
- 2e^{(0)\nu}{}_l \Big[& -\partial^\mu \bar f^{(\rho\sigma)}\partial_\nu \bar f_{(\rho\sigma)} + \frac{1}{2}\partial^\mu \bar f \partial_\nu \bar f - \partial^\sigma \bar f^{(\rho\mu)}\partial_\sigma \bar f_{(\rho\nu)} + \partial^\sigma \bar f^{(\rho\mu)}\partial_\nu \bar f_{(\rho\sigma)} \\
& + \partial^\mu \bar f^{(\rho\sigma)}\partial_\sigma \bar f_{(\rho\nu)} + \frac{1}{2}\partial^\sigma \bar f^{(\rho\mu)}\eta_{\rho\nu}\partial_\sigma \bar f - \frac{1}{2}\partial^\mu \bar f_{(\nu\sigma)}\partial^\sigma \bar f - \partial^\sigma \bar f^{(\rho\mu)}\partial_\sigma f_{[\rho\nu]} \\
& + \partial^\sigma \bar f^{(\rho\mu)}\partial_\nu f_{[\rho\sigma]} + \partial^\mu \bar f^{(\rho\sigma)}\partial_\sigma f_{[\rho\nu]} + \partial^\sigma f^{[\rho\mu]}\partial_\nu \bar f_{(\rho\sigma)} - \partial^\rho f^{[\sigma\mu]}\partial_\sigma \bar f_{(\rho\nu)} \\
& + \frac{1}{2}\partial_\nu f^{[\sigma\mu]}\partial_\sigma \bar f - \partial^\sigma f^{[\rho\mu]}\partial_\nu f_{[\rho\sigma]} - \partial^\rho f^{[\sigma\mu]}\partial_\sigma f_{[\rho\nu]} \\
& - \frac{\kappa}{\lambda}\Big(\partial^\sigma f^{[\rho\mu]}\partial_\sigma \bar f_{(\rho\nu)} - \partial^\mu f^{[\rho\sigma]}\partial_\sigma \bar f_{(\rho\nu)} - \partial^\rho f^{[\sigma\mu]}\partial_\sigma \bar f_{(\rho\nu)} \\
& \quad - \frac{1}{2}\partial^\sigma f^{[\rho\mu]}\eta_{\rho\nu}\partial_\sigma \bar f + \frac{1}{2}\partial^\mu f_{[\nu\sigma]}\partial^\sigma \bar f + \frac{1}{2}\partial_\nu f^{[\sigma\mu]}\partial_\sigma \bar f + \partial^\sigma f^{[\rho\mu]}\partial_\sigma f_{[\rho\nu]} \\
& \quad - 2\partial^\sigma f^{[\rho\mu]}\partial_\nu f_{[\rho\sigma]} - \partial^\mu f^{[\rho\sigma]}\partial_\sigma f_{[\rho\nu]} + \partial^\mu f^{[\rho\sigma]}\partial_\nu f_{[\rho\sigma]} \\
& \quad - \partial^\rho f^{[\sigma\mu]}\partial_\sigma f_{[\rho\nu]}\Big)\Big].
\end{aligned} \tag{3.23}
$$

By using Eqs. (3.19)–(3.22), ${}^G\boldsymbol{T}_l{}^{\mu}$ is found to be given by

$$
\begin{aligned}
{}^G\boldsymbol{T}_l{}^{\mu} = -2e^{(0)\nu}{}_l \Bigg[& \frac{k^\mu k_\nu}{2\kappa}\left(\mathcal{U}^{\rho\sigma}\mathcal{U}_{\rho\sigma}e^{2ik\cdot x} - 2\mathcal{U}^{\rho\sigma}\bar{\mathcal{U}}_{\rho\sigma} + \bar{\mathcal{U}}^{\rho\sigma}\bar{\mathcal{U}}_{\rho\sigma}e^{-2ik\cdot x}\right) \\
& + \frac{k^\mu k_\nu}{2\lambda}\left(\mathcal{V}^{\rho\sigma}\mathcal{V}_{\rho\sigma}e^{2ik\cdot x} - 2\mathcal{V}^{\rho\sigma}\bar{\mathcal{V}}_{\rho\sigma} + \bar{\mathcal{V}}^{\rho\sigma}\bar{\mathcal{V}}_{\rho\sigma}e^{-2ik\cdot x}\right)\Bigg].
\end{aligned} \tag{3.24}
$$

Taking the average of ${}^G\boldsymbol{T}_l{}^{\mu}$ over a space-time region much larger than $|\boldsymbol{k}|^{-1}$, we obtain

$$
\langle {}^G\boldsymbol{T}_l{}^{\mu}\rangle = e^{(0)\nu}{}_l \left[\frac{c^4 k^\mu k_\nu}{2\pi G}\left(|\mathcal{U}_{11}|^2 + |\mathcal{U}_{12}|^2\right) + 4\frac{k^\mu k_\nu}{\lambda}|\mathcal{V}_{12}|^2\right], \tag{3.25}
$$

where we have chosen the direction of the space components $\boldsymbol{k}$ of the four vector k^μ as the third axis. The term in the square brackets of the right-hand side (r.h.s.) of Eq. (3.25) is identical to the corresponding term of the canonical energy-momentum density $\tilde{\boldsymbol{t}}_\nu{}^\mu$ defined by Eq. (A.7) in GR if the antisymmetric part $\mathcal{V}_{12}$ is vanishing.[15]

[15]See, for instance, Section 10.3 of Ref. [15]

4 Quadrupole radiation from point masses

The retarded solutions of Eqs. (3.15) and (3.16) are given by

$$\bar{f}_{(\mu\nu)}(\boldsymbol{x}, x^0) = \frac{\kappa}{4\pi} \int d^3x' \frac{T_{(\mu\nu)}(\boldsymbol{x}', x^0 - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} , \tag{4.1}$$

$$f_{[\mu\nu]}(\boldsymbol{x}, x^0) = \frac{\lambda}{4\pi} \int d^3x' \frac{T_{[\mu\nu]}(\boldsymbol{x}', x^0 - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} , \tag{4.2}$$

respectively. We consider a system of the Newtonian point masses $\{m_a, \xi_a^\mu\}$ ($a = 1, 2, \ldots, N$) as a gravitational wave source, where m_a and ξ_a^μ denote the mass and coordinate of the ath point mass, respectively. For this case, the antisymmetric part of the energy-momentum density becomes

$$T_{[\mu\nu]}(\boldsymbol{x}, x^0) = 0. \tag{4.3}$$

Thus, $f_{[\mu\nu]}$ vanishes, i.e., $f_{[\mu\nu]} = 0$. Applying the retarded expansion to Eq. (4.1) and using the conservation law (3.17), we obtain the quadrupole radiation formula in the rest frame of the system,

$$\bar{f}_{(\alpha\beta)}(\boldsymbol{x}, x^0) = \frac{\kappa}{8\pi r}\partial_0\partial_0 \int d^3x' x'^\alpha x'^\beta T_{(00)}(\boldsymbol{x}', x^0 - r), \tag{4.4a}$$

$$\bar{f}_{(0\alpha)}(\boldsymbol{x}, x^0) = -\frac{x^\beta}{r} \bar{f}_{(\alpha\beta)}(\boldsymbol{x}', x^0 - r), \tag{4.4b}$$

$$\bar{f}_{(00)}(\boldsymbol{x}, x^0) = \frac{\kappa}{4\pi r} \sum_{a=1}^{N} m_a c^2 + \frac{x^\alpha x^\beta}{r^2} \bar{f}_{(\alpha\beta)}(\boldsymbol{x}', x^0 - r), \tag{4.4c}$$

with $r = (x^\alpha x^\alpha)^{\frac{1}{2}}$.[16] Since we regard each velocity of the mass points to be much smaller than the velocity of light, we can use the quantity

$$T_{(00)}(\boldsymbol{x}, x^0) = \sum_{a=1}^{N} m_a c^2 \delta^{(3)}(\boldsymbol{x} - \boldsymbol{\xi}_a(t)) \tag{4.5}$$

[16]In our convention, letters from the beginning of the Greek alphabet, α, β, γ, $\cdots$, and those from the beginning of the Latin alphabet, a, b, c, $\cdots$, take the values 1, 2, and 3, unless otherwise stated. Also, here we have used the usual summation convention for repeated indices.

as the energy density of the system, with $t \stackrel{\rm def}{=} x^0/c$. Substituting Eq. (4.5) into Eq. (4.4), we obtain

$$\bar{f}_{(\alpha\beta)} = \frac{\kappa}{8\pi r}\ddot{D}_{\alpha\beta} \ , \quad \bar{f}_{(0\alpha)} = -\frac{x^\beta}{r}\frac{\kappa}{8\pi r}\ddot{D}_{\alpha\beta} \ , \tag{4.6a}$$

$$\bar{f}_{(00)} = \frac{\kappa}{4\pi r}\sum_{a=1}^{N} m_a c^2 + \frac{x^\alpha x^\beta}{r^2}\frac{\kappa}{8\pi r}\ddot{D}_{\alpha\beta} \ . \tag{4.6b}$$

Here, $D_{\alpha\beta}$ is the quadrupole moment of the mass distribution defined by

$$D_{\alpha\beta} \stackrel{\rm def}{=} \sum_{a=1}^{N} m_a \xi_a^\alpha \xi_a^\beta \ , \tag{4.7}$$

where we have defined $\ddot{D}_{\alpha\beta} \stackrel{\rm def}{=} \partial^2 D_{\alpha\beta}/\partial^2 t$. The wave form given by Eqs. (4.6) and (4.7) is the same as the corresponding wave form in GR, which is consistent with the result given in Ref. [11].

5 Emission rates of energy-momentum and angular momentum

In this section, we examine time averages of emission rates of the energy-momentum and angular momentum carried by the quadrupole radiation given by Eq. (4.6).

In ENGR, for the asymptotic conditions (2.45) and (2.46), we have four quantities that are conserved as long as they are finite, the dynamical energy-momentum M_k, "spin" angular momentum S_{kl}, canonical energy-momentum $M^c{}_\mu$, and "extended orbital angular momentum" $L_\mu{}^\nu$ [5]. *The dynamical energy momentum does not depend on the asymptotic form of the Higgs-type field* ψ^k *explicitly,* but the quantities $S_{kl}, M^c{}_\mu$ and $L_\mu{}^\nu$ depend on the asymptotic form. With this in mind, following Ref. [5], we examine the case in which the asymptotic form of the Higgs-type field is

$$\psi^k \simeq e^{(0)k}{}_\mu x^\mu + \psi^{(0)k} \ ,$$

with $\psi^{(0)k}$ constant. We also consider the slightly generalized case in which

$$\psi^k \simeq \rho e^{(0)k}{}_\mu x^\mu + \psi^{(0)k}.$$

The asymptotic form of the function ρ is determined in §5.2.

5.1 The case $\psi^k \simeq e^{(0)k}{}_{\mu} x^{\mu} + \psi^{(0)k}$

5.1.1 Dynamical energy-momentum loss

In order to evaluate the emission rate of the dynamical energy-momentum, we integrate the differential conservation law (2.37) over a large solid sphere V with radius r, yielding

$$\partial_0 \int_V {}^{\rm tot}\boldsymbol{T}_k{}^0 d^3x = -\int_S {}^{\rm tot}\boldsymbol{T}_k{}^{\alpha} r^2 n^{\alpha} d\Omega \ , \tag{5.1}$$

where S and $d\Omega$ represent the two-dimensional surface of V and the differential solid angle, respectively. Also, n^{α} stands for the unit radial vector defined by $n^{\alpha} \stackrel{\rm def}{=} x^{\alpha}/r$. Taking into account the fact that the energy-momentum density of point masses vanishes for very large r, we can rewrite the r.h.s. of Eq. (5.1) as

$$-\int_S {}^{\rm tot}\boldsymbol{T}_k{}^{\alpha} r^2 n^{\alpha} d\Omega = -\int_S {}^{G}\boldsymbol{T}_k{}^{\alpha} r^2 n^{\alpha} d\Omega \ . \tag{5.2}$$

The density ${}^{G}\boldsymbol{T}_k{}^{\mu}$ takes, up to terms of order $O(1/r^2)$, the form

$${}^{G}\boldsymbol{T}_k{}^{\mu} = e^{(0)\nu}{}_k \, {}^{G}T^{(0)}{}_{\nu}{}^{\mu} \ , \tag{5.3}$$

with

$${}^{G}T^{(0)}{}_{\nu}{}^{\mu} \stackrel{\rm def}{=} \frac{1}{\kappa}\left(\partial^{\mu}\bar{f}^{(\rho\sigma)}\partial_{\nu}\bar{f}_{(\rho\sigma)} - \frac{1}{2}\partial^{\mu}\bar{f}\partial_{\nu}\bar{f}\right) . \tag{5.4}$$

Using the solution (4.6), and averaging over one period of motion of the system of point masses, we obtain

$$-\left\langle \frac{dE}{dt} \right\rangle = \frac{G}{5c^5}\left\langle \dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\beta} - \frac{1}{3}\dddot{D}_{\alpha\alpha}\dddot{D}_{\beta\beta} \right\rangle , \tag{5.5}$$

for a total energy $E \stackrel{\rm def}{=} -M_{(0)}$ of the system, where $\langle \cdots \rangle$ denotes the operation of averaging over one period of motion of the system of the point masses, and we have set $e^{(0)k}{}_{\mu} = \delta^k{}_{\mu}$ for simplicity. Also, we obtain

$$\frac{dM_a}{dt} = 0. \tag{5.6}$$

It is worth mentioning that the quantity ${}^{G}T^{(0)}{}_{\mu}{}^{\nu}$ is identical to the r.h.s. of Eq. (A.14), and the time average of the emission rate of the dynamical energy-momentum, which is given by Eqs. (5.5) and (5.6), is identical to that of the canonical energy-momentum in GR.

5.1.2 "Spin" angular momentum loss

The density ${}^{G}\boldsymbol{S}_{kl}{}^{\mu}$ has the expression

$$
\begin{aligned}
{}^{G}\boldsymbol{S}_{kl}{}^{\mu} &= 2\eta_{km}\eta_{ln}e^{(0)m}{}_{[\rho}e^{(0)n}{}_{\sigma]}\psi^{\rho}(-g)t^{\sigma\mu}_{\mathrm{LL}} \\
&\quad - 2e^{(0)\nu}{}_{[k}\eta_{l]m}e^{(0)m}{}_{\tau}\left[Z^{(1)}{}_{\nu}{}^{\mu}\psi^{\tau} + \eta_{\nu\sigma}(Z^{(2)\mu\sigma\tau} - Z^{(3)\mu\lambda\sigma}\psi^{\tau}{}_{,\lambda})\right] ,
\end{aligned}
\tag{5.7}
$$

where we have defined

$$
\psi^{\rho} \stackrel{\mathrm{def}}{=} e^{(0)\rho}{}_{k}\psi^{k} , \tag{5.8}
$$

$$
\begin{aligned}
\kappa Z^{(1)}{}_{\nu}{}^{\mu} &\stackrel{\mathrm{def}}{=} -\frac{1}{2}\delta_{\nu}{}^{\mu}\partial^{\sigma}\bar{f}^{(\lambda\rho)}\partial_{\rho}\bar{f}_{(\lambda\sigma)} - \partial^{\sigma}\bar{f}^{(\rho\mu)}\partial_{\sigma}\bar{f}_{(\rho\nu)} + \partial^{\sigma}\bar{f}^{(\rho\mu)}\partial_{\nu}\bar{f}_{(\rho\sigma)} \\
&\quad + \partial^{\mu}\bar{f}^{(\rho\sigma)}\partial_{\sigma}\bar{f}_{(\rho\nu)} - \frac{1}{2}\partial^{\sigma}\bar{f}^{(\rho\mu)}\eta_{\rho\nu}\partial_{\sigma}\bar{f} + \frac{1}{2}\partial^{\mu}\bar{f}_{(\nu\sigma)}\partial^{\sigma}\bar{f} ,
\end{aligned}
\tag{5.9}
$$

$$
\kappa Z^{(2)\mu\sigma\tau} \stackrel{\mathrm{def}}{=} \left(\partial^{\lambda}\bar{f}^{(\sigma\mu)} - \partial^{\mu}\bar{f}^{(\sigma\lambda)}\right)\left(\delta^{\tau}{}_{\lambda} + \eta^{\tau\rho}\bar{f}_{(\rho\lambda)} - \frac{1}{2}\delta^{\tau}{}_{\lambda}\bar{f}\right), \tag{5.10}
$$

$$
\kappa Z^{(3)\mu\lambda\sigma} \stackrel{\mathrm{def}}{=} \partial^{\lambda}\bar{f}^{(\sigma\mu)} - \partial^{\mu}\bar{f}^{(\sigma\lambda)} . \tag{5.11}
$$

Also, $t^{\sigma\mu}_{\mathrm{LL}}$ denotes the energy-momentum density introduced by Landau and Lifshitz, whose explicit form is given in Appendix A. Using a method similar to that employed in §5.1.1, we obtain

$$
\partial_0 \int_V {}^{\mathrm{tot}}\boldsymbol{S}_{kl}{}^{0}d^3x = -\int_S {}^{G}\boldsymbol{S}_{kl}{}^{\alpha}r^2 n^{\alpha}d\Omega . \tag{5.12}
$$

In order to estimate the r.h.s. of Eq. (5.12), we set

$$
\psi^{\mu} = x^{\mu} + \psi^{(0)\mu} + \tilde{\psi}^{\mu} , \tag{5.13}
$$

where $\psi^{(0)\mu}$ is a constant. From Eqs. (4.6) and (5.7)–(5.13), we can show that[17]

$$
\begin{aligned}
-\left\langle \frac{dS_{(0)a}}{dt} \right\rangle &= -\psi^{(0)a}\frac{G}{5c^6}\left\langle \dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\beta} - \frac{1}{3}\dddot{D}_{\alpha\alpha}\dddot{D}_{\beta\beta} \right\rangle \\
&= -2\psi^{(0)}{}_{[a}\left\langle \frac{dM_{(0)]}}{dt} \right\rangle
\end{aligned}
\tag{5.14}
$$

[17]Note that we have distinguished between Latin and Greek indices here. (See footnote 9 on page 8.)

if

$$\lim_{r\to\infty}\tilde{\psi}^{\mu}=0,\quad \lim_{r\to\infty}\tilde{\psi}^{\mu}{}_{,0}=0,\quad \lim_{r\to\infty}r\tilde{\psi}^{\mu}{}_{,\beta}=0, \tag{5.15}$$

where again we have set $e^{(0)k}{}_{\mu}=\delta^{k}{}_{\mu}$. Also, we have

$$-\left\langle\frac{dS_{ab}}{dt}\right\rangle=\frac{2G}{5c^5}\left\langle\ddot{D}_{a\gamma}\dddot{D}_{b\gamma}-\ddot{D}_{b\gamma}\dddot{D}_{a\gamma}\right\rangle \tag{5.16}$$

if

$$\lim_{r\to\infty}\tilde{\psi}^{\alpha}=0,\quad \lim_{r\to\infty}\tilde{\psi}^{\alpha}{}_{,0}=0,\quad \lim_{r\to\infty}r\tilde{\psi}^{\alpha}{}_{,\beta}=0. \tag{5.17}$$

The time average of the emission rate of the "spin" angular momentum S_{ab} is the same as that of the space-space component of the angular momentum in GR.

5.1.3 Canonical energy-momentum and orbital angular momentum losses

In the weak-field approximation, the canonical energy-momentum density of the gravitational field ${}^{G}\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}$ becomes

$${}^{G}\widetilde{\boldsymbol{T}}_{\mu}{}^{\nu}=\eta_{\rho\sigma}Z^{(3)\nu\lambda\sigma}\psi^{\rho}{}_{,\lambda\mu}-{}^{G}T^{(0)}{}_{\lambda}{}^{\nu}\psi^{\lambda}{}_{,\mu}\ . \tag{5.18}$$

Then, using Eqs. (4.6), (5.11) and (5.13), we can show that

$$\frac{dM^{c}{}_{\mu}}{dt}=0 \tag{5.19}$$

if the conditions

$$\lim_{r\to\infty}\tilde{\psi}^{\mu}{}_{,\nu}=0,\quad \lim_{r\to\infty}\tilde{\psi}^{\mu}{}_{,00}=0,\quad \lim_{r\to\infty}r\tilde{\psi}^{\mu}{}_{,\beta\nu}=0 \tag{5.20}$$

are satisfied.

Finally, we examine the "extended orbital angular momentum." In the weak-field approximation, the "extended orbital angular momentum" density of the gravitational field ${}^{G}\widetilde{\boldsymbol{M}}_{\mu}{}^{\nu\lambda}$ becomes

$${}^{G}\widetilde{\boldsymbol{M}}_{\mu}{}^{\nu\lambda}=2Z^{(4)}{}_{\mu}{}^{\nu\lambda}-2Z^{(3)\lambda\nu\sigma}\eta_{\sigma\rho}\psi^{\rho}{}_{,\mu}-2x^{\nu}Z^{(3)\lambda\tau\sigma}\eta_{\rho\sigma}\psi^{\rho}{}_{,\tau\mu}+2x^{\nu}Z^{(5)}{}_{\sigma}{}^{\lambda}\psi^{\sigma}{}_{,\mu}\ , \tag{5.21}$$

where we have defined

$$\kappa Z^{(4)}{}_{\mu}{}^{\nu\lambda} \stackrel{\text{def}}{=} \left(\partial^{\nu}\bar{f}^{(\sigma\lambda)} - \partial^{\lambda}\bar{f}^{(\sigma\nu)}\right)\left(\eta_{\sigma\mu} + \bar{f}_{(\sigma\mu)} - \frac{1}{2}\eta_{\sigma\mu}\bar{f}\right)$$
$$+ x^{\nu}\Bigg[\delta_{\mu}{}^{\lambda}\left(\frac{1}{2}\partial^{\sigma}\bar{f}^{(\tau\rho)}\partial_{\sigma}\bar{f}_{(\tau\rho)} - \frac{1}{2}\partial^{\sigma}\bar{f}^{(\tau\rho)}\partial_{\rho}\bar{f}_{(\tau\sigma)} - \frac{1}{4}\partial^{\sigma}\bar{f}\partial_{\sigma}\bar{f}\right)$$
$$+ \left(\partial^{\rho}\bar{f}^{(\sigma\lambda)} - \partial^{\lambda}\bar{f}^{(\sigma\rho)}\right)\left(\partial_{\mu}\bar{f}_{(\sigma\rho)} - \frac{1}{2}\eta_{\sigma\rho}\partial_{\mu}\bar{f}\right)\Bigg], \tag{5.22}$$

$$\kappa Z^{(5)}{}_{\sigma}{}^{\lambda} \stackrel{\text{def}}{=} \delta_{\sigma}{}^{\lambda}\left(-\frac{1}{2}\partial^{\xi}\bar{f}^{(\tau\rho)}\partial_{\xi}\bar{f}_{(\tau\rho)} + \frac{1}{2}\partial^{\xi}\bar{f}^{(\tau\rho)}\partial_{\rho}\bar{f}_{(\tau\xi)} + \frac{1}{4}\partial^{\rho}\bar{f}\partial_{\rho}\bar{f}\right)$$
$$+ \partial^{\lambda}\bar{f}^{(\rho\tau)}\partial_{\sigma}\bar{f}_{(\rho\tau)} - \frac{1}{2}\partial^{\lambda}\bar{f}\partial_{\sigma}\bar{f} + \partial^{\tau}\bar{f}^{(\rho\lambda)}\partial_{\tau}\bar{f}_{(\rho\sigma)} - \partial^{\tau}\bar{f}^{(\rho\lambda)}\partial_{\sigma}\bar{f}_{(\rho\tau)}$$
$$- \partial^{\lambda}\bar{f}^{(\rho\tau)}\partial_{\tau}\bar{f}_{(\rho\sigma)} - \frac{1}{2}\partial^{\tau}\bar{f}^{(\rho\lambda)}\eta_{\rho\sigma}\partial_{\tau}\bar{f} + \frac{1}{2}\partial^{\lambda}\bar{f}_{(\sigma\tau)}\partial^{\tau}\bar{f}\ . \tag{5.23}$$

As can be shown by using Eqs. (4.6), (5.13), (5.21), (5.22) and (5.23), we have the relations

$$\frac{dL_{\mu}{}^{0}}{dt} = 0, \quad \frac{dL_{0}{}^{\alpha}}{dt} = 0, \tag{5.24}$$

$$\left\langle\frac{dL_{\alpha}{}^{\beta}}{dt}\right\rangle = -\frac{G}{3c^5}\left\langle\ddot{D}_{\alpha\gamma}\dddot{D}_{\beta\gamma} + \ddot{D}_{\alpha\gamma}\dddot{D}_{\beta\gamma} - \ddot{D}_{\gamma\gamma}\dddot{D}_{\alpha\beta}\right\rangle \tag{5.25}$$

if the conditions

$$\lim_{r\to\infty} r\tilde{\psi}^{\mu}{}_{,\nu} = 0, \quad \lim_{r\to\infty} r\tilde{\psi}^{\mu}{}_{,00} = 0, \quad \lim_{r\to\infty} r^2\tilde{\psi}^{\mu}{}_{,\beta\nu} = 0 \tag{5.26}$$

are satisfied. The emission rate of the component $L_{\alpha}{}^{\beta}$ is finite. However, for the antisymmetric part of $L_{\alpha\beta}$, which is the three-dimensional orbital angular momentum, we have

$$\frac{dL_{[\alpha\beta]}}{dt} = 0. \tag{5.27}$$

To summarize, the emission rates $dM^{c}{}_{\mu}/dt$ and $dL_{[\mu\nu]}/dt$ are both vanishing if the condition (5.26) is satisfied.[18]

5.2 The case $\psi^k \simeq \rho e^{(0)k}{}_{\mu}x^{\mu} + \psi^{(0)k}$

Also in this case, the expressions (5.5) and (5.6) for the time average of the emission rate of the dynamical energy-momentum hold, as seen from Eq. (5.3).

[18] Note that $L_{[\mu\nu]}$ corresponds to the generator of the Lorentz coordinate transformations and that it possibly represents the four-dimensional orbital angular momentum.

We express ψ^μ as

$$\psi^\mu = \rho(r,t)x^\mu + \psi^{(0)\mu} + \tilde{\psi}^\mu \ . \tag{5.28}$$

Then, the expression (5.14) for the time average of the emission rate of the time-space component of the "spin" angular momentum holds if the condition (5.15) is satisfied by $\tilde{\psi}^\mu$ in Eq. (5.28). For the space-space component S_{ab}, using Eqs. (4.6) and (5.7)–(5.12), we find

$$-\left\langle \frac{dS_{ab}}{dt} \right\rangle = \frac{2G}{5c^5}\frac{3+\rho_c}{4}\left\langle \ddot{D}_{a\gamma}\dddot{D}_{b\gamma} - \ddot{D}_{b\gamma}\dddot{D}_{a\gamma} \right\rangle \tag{5.29}$$

if the condition (5.17) and

$$\lim_{r\to\infty} \rho = \rho_c \ , \tag{5.30}$$

with ρ_c constant, are satisfied by ρ and $\tilde{\psi}^\mu$ in Eq. (5.28).

From Eqs. (4.6), (5.4), (5.11), (5.18) and (5.28), we have the relations

$$\left\langle \frac{dM^c{}_0}{dt} \right\rangle = \frac{G}{5c^5}(1-\rho_c)\left\langle \dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\beta} - \frac{1}{3}\dddot{D}_{\alpha\alpha}\dddot{D}_{\beta\beta} \right\rangle , \tag{5.31}$$

$$\frac{dM^c{}_\alpha}{dt} = 0 \tag{5.32}$$

if the condition (5.20) and

$$\lim_{r\to\infty} \rho = \rho_c \ , \quad \lim_{r\to\infty} r\rho_{,0} = 0, \quad \lim_{r\to\infty} r\rho_{,00} = 0 \tag{5.33}$$

are satisfied.

Under the condition (5.30), the time average of the emission rate of the "extended orbital angular momentum" diverges. However, using Eqs. (4.6), (5.21), (5.22), (5.23) and (5.28), we can show that

$$\frac{dL_{[0\alpha]}}{dt} = 0, \tag{5.34}$$

$$-\left\langle \frac{dL_{[\alpha\beta]}}{dt} \right\rangle = \frac{2G}{5c^5}\frac{1-\rho_c}{4}\left\langle \ddot{D}_{a\gamma}\dddot{D}_{b\gamma} - \ddot{D}_{b\gamma}\dddot{D}_{a\gamma} \right\rangle \tag{5.35}$$

if the conditions (5.26) and (5.30) are satisfied.

For a space-time satisfying the asymptotic conditions (2.45) and (2.46), the sum $S_{kl} + e^{(0)\mu}{}_k e^{(0)\nu}{}_l L_{[\mu\nu]}$ is well-defined and conserved for the case $\psi^k \simeq \rho_c e^{(0)k}{}_\mu x^\mu +$

$\psi^{(0)k}$ ($\rho_c \neq 1$), as described in Ref. [5]. In the case under consideration, we have

$$-\left\langle \frac{d}{dt}(S_{(0)a} + L_{[0a]}) \right\rangle = -2\psi^{(0)}{}_{[a} \left\langle \frac{dM_{(0)]}}{dt} \right\rangle , \tag{5.36}$$

$$-\left\langle \frac{d}{dt}\left(S_{ab} + L_{[ab]}\right) \right\rangle = \frac{2G}{5c^5} \left\langle \ddot{D}_{a\gamma}\dddot{D}_{b\gamma} - \ddot{D}_{b\gamma}\dddot{D}_{a\gamma} \right\rangle \tag{5.37}$$

if the conditions

$$\lim_{r\to\infty} \tilde{\psi}^{\mu} = 0, \quad \lim_{r\to\infty} r\tilde{\psi}^{\mu}{}_{,\nu} = 0, \quad \lim_{r\to\infty} r\tilde{\psi}^{\mu}{}_{,00} = 0, \quad \lim_{r\to\infty} r^2\tilde{\psi}^{\mu}{}_{,\beta\nu} = 0 \tag{5.38}$$

are satisfied. Here, again, we have set $e^{(0)\mu}{}_{k} = \delta^{\mu}{}_{k}$.

6 Summary and discussion

In an extended, new form of general relativity, we have examined energy-momentum and angular momentum carried by gravitational wave radiated from a system of Newtonian point masses in a weak-field approximation. The results are summarized as follows.

1. The form of the gravitational wave is identical to the corresponding wave form in GR, which is consistent with the result in Ref. [11].

2. The average value of $\langle {}^{G}\boldsymbol{T}_{l}{}^{\mu} \rangle$ for the dynamical energy-momentum of a plane wave is obtained from Eq. (3.25), and it is the same as that of the corresponding canonical energy-momentum in GR [15].

3. The dynamical energy-momentum density ${}^{G}\boldsymbol{T}_{k}{}^{\mu}$ takes, up to order $O(1/r^2)$, the form given by Eq. (5.3) with Eq. (5.4), and this is essentially the same as the corresponding expression (A.14) for the canonical energy-momentum density in GR, and the time average of the energy-momentum emission rate is given by Eqs. (5.5) and (5.6), which is identical to that in GR.

4. The time average of the emission rate of the "spin" angular momentum is given by Eqs. (5.14) and (5.16) if the conditions (5.15) and (5.17) for the form (5.13) of the Higgs-type field are satisfied. The expression (5.16) is the same as the corresponding expression for the angular momentum in GR.

5. The emission rates of both the canonical energy-momentum and the orbital angular momentum vanish if the conditions (5.20) and (5.26) for ψ^k given by the expression (5.13) are both satisfied.

6. Under the condition (5.30) for the form (5.28) of the Higgs-type field, the time average of the emission rates of the "spin" angular momentum and the canonical energy-momentum depend on the constant ρ_c. Moreover, the time average of the emission rate of the "extended orbital angular momentum" diverges. However, the time average of the emission rate of the sum [5] $S_{kl} + e^{(0)\mu}{}_k e^{(0)\nu}{}_l L_{[\mu\nu]}$ is finite, and its space-space component is identical to the corresponding expression for the angular momentum in GR if the condition (5.38) is satisfied.

Finally, we would like to add the following:

A. As we have stated repeatedly, the dynamical energy-momentum and "spin" angular momentum are generators of *internal* translations and *internal* $SL(2, C)$ transformations. The former does not depend on the non-dynamical field ψ^k explicitly, but the latter does. For vierbeins possessing asymptotic forms satisfying Eqs. (2.45) and (2.46), they give the total energy-momentum and *total* (=*spin*+*orbital*) angular momentum of the system when the field ψ^k is chosen as $\psi^k = e^{(0)k}{}_\mu x^\mu + \psi^{(0)k} + O(1/r^\beta)$. The generator of the affine *coordinate* transformations, contrastingly, vanishes. The results summarized in 2–5 above are consistent with this. The discussion in Appendix B gives further support to the choice of the asymptotic form of ψ^k given by Eq. (5.13) satisfying the condition (5.17).

B. The "spin" angular momentum depends on the Higgs-type field ψ^k, and it is meaningful if ψ^k satisfies a suitable condition requiring this field to be the same as the Minkowskian coordinates on the boundary of a sphere of infinite radius, as is known from §5 and from Refs. [5, 7, 8, 9]. The field ψ^k behaves like a Minkowskian coordinate system under *internal* $\overline{\text{Poincaré}}$ transformations, and its existence is a necessary consequence of the structure of the group $\bar{P}_0$ and a basic postulate regarding the space-time. Also, this field is closely related to

the existence of the spinor structure [4]. However, the physical and geometrical meaning of this field has not yet been fully clarified.

C. In considering the energy-momentum and angular momentum, there are two possibilities in choosing the set of independent field variables [5, 6], i.e., the set $\{\psi^k, A^k{}_\mu, \phi^A\}$ and the set $\{\psi^k, e^k{}_\mu, \phi^A\}$. In the present paper, we have employed $\{\psi^k, A^k{}_\mu, \phi^A\}$ as the set of independent field variables, because this choice is superior to the other, as shown Refs. [5] and [6].[19]

D. (a) The asymptotic conditions (5.26) and (5.30) for the field ψ^k are stronger than the corresponding conditions for vierbeins whose asymptotic forms satisfy Eqs. (2.45) and (2.46) [5]. This is natural, because vierbeins in which gravitational waves propagate do not satisfy the conditions (2.45) and (2.46) in general.

   (b) For the form (5.28) of a Higgs-type field, the time average of the emission rate of the canonical energy, which is given by Eq. (5.31), is identical to the corresponding expression in GR if the condition (5.33) with $\rho_c = 0$ is satisfied. However, when $\rho_c = 0$, neither the time average of the emission rate for the space-space component of the "spin" angular momentum S_{kl} nor that of the orbital angular momentum $L_{[\mu\nu]}$ are the same as the corresponding expression in GR. The time average of the emission rate for the space-space component of the sum $S_{kl} + e^{(0)\mu}{}_k e^{(0)\nu}{}_l L_{[\mu\nu]}$ is the same as the corresponding expression in GR. However, the sum $S_{kl} + e^{(0)\mu}{}_k e^{(0)\nu}{}_l L_{[\mu\nu]}$ is an artificial quantity, because the "spin" angular momentum S_{kl} and orbital angular momentum $L_{[\mu\nu]}$ are associated with transformations that differ with each other.

E. The transformation property of the dynamical energy-momentum density of gravity in ENGR is different from that of the canonical energy-momentum density of gravity in GR. The former behaves as a vector density under general

[19] It should be noted that when $\{\psi^k, A^k{}_\mu, \phi^A\}$ is employed as the set of independent field variables, the Lagrangian L is considered to have an explicit ψ^k dependence, because L is a function of $e^k{}_\mu = \psi^k{}_{,\mu} + A^k{}_\mu$, ϕ^A and their derivatives.

coordinate transformations, while the latter is not tensorial. Nevertheless, the two densities take the same form up to order $O(1/r^2)$.

F. In ENGR, the world line of a macroscopic test body is the geodesic of the metric g, as in the case of GR,[20] and hence ENGR with the condition (3.12) accurately describes the variation of the period of motion of the binary pulsar PSR1913+16 [16]. This is consistent with results in Refs. [10] and [11].

G. As far as the gravitational radiation from Newtonian point masses treated in the weak-field approximation is concerned, the losses of energy-momentum and angular momentum as well as the wave form are independent of the parameter c_3 in the gravitational Lagrangian density, and they are identical to the corresponding quantities in GR. The effects of c_3 reveal themselves at higher orders.

H. We have considered the case of a weak field under the condition (3.12), but the field equations (3.4) and (3.5) can be solved also in the case with the following [17]:

$$c_1 + c_2 \neq 0, \quad c_1 - \frac{4}{9} c_3 \neq 0. \tag{6.1}$$

The dynamical structure of the system with the condition (6.1) is significantly different from that with the condition (3.12). Although the values of the parameters c_1 and c_2 are severely restricted as

$$c_1 \simeq -\frac{1}{3\kappa} \simeq -c_2 \tag{6.2}$$

by the results of Solar System experiments [13], there still remains the possibility that Eq. (3.12) is not satisfied. The gravitational radiation for the case in which the parameters c_1, c_2 and c_3 satisfy Eq. (6.1) is also worth examining.

From A, C and D, we deduce the following: The choice $\{\psi^k, A^k{}_\mu, \phi^A\}$ with Eq. (5.13) satisfying the conditions (5.15) and (5.17) is superior to the other possible choices, and the generator of *internal* $\overline{\text{Poincaré}}$ transformations accurately describe

[20] Note that the behavior of the Dirac particle in ENGR differs from that in GR. (See Ref. [13].)

the energy-momentum and angular momentum for a wide class of gravitating systems, including space-times in which there are gravitational waves.

In the teleparallel theory of gravity, there have been several attempts [18, 19, 20, 21, 22, 23] to define well-behaved energy-momentum and angular momentum densities. For the case of the teleparallel equivalent of general relativity, i.e. the case with the condition (2.27) in our notation, this problem is studied in Refs. [18, 19, 20]. The gravitational energy-momentum density $hj_a{}^\rho$ in Ref. [18] is the same as our ${}^G\boldsymbol{T}_k{}^\mu$. In Refs. [19] and [20], Hamiltonian formalism is developed, and a natural definition of the energy-momentum density of the gravitational field is given. In addition, the angular-momentum density is examined in Ref. [19]. In Ref. [21], an energy-momentum current that transforms as a tensor under diffeomorphisms of the space-time manifold and under global $SO(1,3)$ transformations is proposed in a co-frame field formulation of the general teleparallel theory of gravity. In Refs. [22] and [23], the energy-momentum and angular momentum densities for general teleparallel theory of an isolated gravitating system are examined.

The energy-momentum and angular momentum for gravitational waves, however, are not examined in Refs. [18, 19, 20, 21, 22, 23], and in Refs. [19, 20, 21, 22, 23], the energy-momentum and angular momentum are not related to the generators of *internal* Poincaré transformations, and there appears no field that corresponds to our ψ^k. It is worth examining the relation between their [19, 20, 21, 22, 23] energy-momentum and angular momentum densities and ours.

Acknowledgements

The authors would like to express their sincere thanks to Professor K. Nakao for valuable discussions.

A Linearized Einstein Theory

For convenience, we give here a short summary of linearized GR.

The gravitational Lagrangian density is given by

$$\boldsymbol{L}_{\rm GR} \stackrel{\rm def}{=} \frac{1}{2\kappa}\sqrt{-g}g^{\mu\nu}\left[\left\{ {\lambda \atop \mu\ \rho} \right\}\left\{ {\rho \atop \nu\ \lambda} \right\} - \left\{ {\lambda \atop \mu\ \nu} \right\}\left\{ {\rho \atop \lambda\ \rho} \right\}\right], \tag{A.1}$$

where we have defined the Christoffel symbols by

$$\left\{ {\lambda \atop \mu\ \nu} \right\} \stackrel{\rm def}{=} \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \tag{A.2}$$

The Einstein equation takes the form

$$R_{\mu\nu}(\{\}) - \frac{1}{2}g_{\mu\nu}R(\{\}) = \kappa T_{\mu\nu}\ , \tag{A.3}$$

where we have defined the Riemann-Christoffel curvature, Ricci tensor and scalar curvature by

$$R^\rho{}_{\sigma\mu\nu}(\{\}) \stackrel{\rm def}{=} \partial_\mu\left\{ {\rho \atop \sigma\ \nu} \right\} - \partial_\nu\left\{ {\rho \atop \sigma\ \mu} \right\} + \left\{ {\rho \atop \lambda\ \mu} \right\}\left\{ {\lambda \atop \sigma\ \nu} \right\} - \left\{ {\rho \atop \lambda\ \nu} \right\}\left\{ {\lambda \atop \sigma\ \mu} \right\}, \tag{A.4}$$

$$R_{\mu\nu}(\{\}) \stackrel{\rm def}{=} R^\rho{}_{\mu\rho\nu}\ , \tag{A.5}$$

$$R(\{\}) \stackrel{\rm def}{=} g^{\mu\nu}R_{\mu\nu}(\{\}), \tag{A.6}$$

respectively. Also, $T_{\mu\nu}$ denotes the energy-momentum density of the gravitational source. The canonical energy-momentum density is defined by

$$\tilde{\boldsymbol{t}}_\mu{}^\nu \stackrel{\rm def}{=} \delta_\mu{}^\nu \boldsymbol{L}_{\rm GR} - \frac{\partial \boldsymbol{L}_{\rm GR}}{\partial g_{\rho\sigma,\nu}}g_{\rho\sigma,\mu}\ . \tag{A.7}$$

We consider a metric perturbation $h_{\mu\nu}$ from Minkowskian space-time, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \quad |h_{\mu\nu}| \ll 1. \tag{A.8}$$

The linearized field equations are given by the form in Eq. (3.15), up to an overall factor of 2,

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}\ , \tag{A.9}$$

with the harmonic coordinate condition

$$\partial_\nu \bar{h}^{\mu\nu} = 0, \tag{A.10}$$

where we have defined

$$\bar{h}_{\mu\nu} \stackrel{\mathrm{def}}{=} h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \ , \quad h \stackrel{\mathrm{def}}{=} \eta^{\mu\nu} h_{\mu\nu} \ , \tag{A.11}$$

$$\bar{h} \stackrel{\mathrm{def}}{=} \eta^{\mu\nu} \bar{h}_{\mu\nu} \ . \tag{A.12}$$

Note that the perturbation $\bar{h}_{\mu\nu}$ corresponds to $2\bar{f}_{(\mu\nu)}$ in §3. At lowest order, $\tilde{\boldsymbol{t}}_{\mu}{}^{\nu}$ becomes

$$\begin{aligned} 2\kappa\, \tilde{\boldsymbol{t}}_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} & \left(\frac{1}{2}\partial^{\rho}\bar{h}^{\lambda\sigma}\partial_{\lambda}\bar{h}_{\rho\sigma} - \frac{1}{4}\partial^{\lambda}\bar{h}^{\rho\sigma}\partial_{\lambda}\bar{h}_{\rho\sigma} + \frac{1}{8}\partial^{\sigma}\bar{h}\partial_{\sigma}\bar{h} \right) \\ & + \frac{1}{2}\partial_{\mu}\bar{h}_{\rho\sigma}\partial^{\nu}\bar{h}^{\rho\sigma} - \frac{1}{4}\partial_{\mu}\bar{h}\partial^{\nu}\bar{h} - \partial_{\mu}\bar{h}_{\rho\sigma}\partial^{\rho}\bar{h}^{\sigma\nu} \ , \end{aligned} \tag{A.13}$$

which is approximated, up to order $O(1/r^2)$, as

$$\tilde{\boldsymbol{t}}_{\mu}{}^{\nu} = \frac{1}{4\kappa}\left(\partial_{\mu}\bar{h}_{\rho\sigma}\partial^{\nu}\bar{h}^{\rho\sigma} - \frac{1}{2}\partial_{\mu}\bar{h}\partial^{\nu}\bar{h} \right) . \tag{A.14}$$

Finally, the energy-momentum density of gravity proposed by Landau and Lifshitz [14] is defined by

$$(-g)t_{\mathrm{LL}}^{\mu\nu} \stackrel{\mathrm{def}}{=} \theta^{\mu\nu} - (-g)T^{\mu\nu} \ , \tag{A.15}$$

where $\theta^{\mu\nu}$ is the symmetric energy-momentum density defined by Eq. (2.51). By using Eqs. (2.51) and (A.3), we obtain from Eq. (A.15) the following:

$$\begin{aligned} 2\kappa\, t_{\mathrm{LL}}^{\mu\nu} = & \left(2\begin{Bmatrix} \sigma \\ \lambda\ \rho \end{Bmatrix}\begin{Bmatrix} \tau \\ \sigma\ \tau \end{Bmatrix} - \begin{Bmatrix} \sigma \\ \lambda\ \tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \rho\ \sigma \end{Bmatrix} - \begin{Bmatrix} \sigma \\ \lambda\ \sigma \end{Bmatrix}\begin{Bmatrix} \tau \\ \rho\ \tau \end{Bmatrix} \right)(g^{\mu\lambda}g^{\nu\rho} - g^{\mu\nu}g^{\lambda\rho}) \\ & + g^{\mu\lambda}g^{\rho\sigma}\left(\begin{Bmatrix} \nu \\ \lambda\ \tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \rho\ \sigma \end{Bmatrix} + \begin{Bmatrix} \nu \\ \rho\ \sigma \end{Bmatrix}\begin{Bmatrix} \tau \\ \lambda\ \tau \end{Bmatrix} - \begin{Bmatrix} \nu \\ \sigma\ \tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \lambda\ \rho \end{Bmatrix} - \begin{Bmatrix} \nu \\ \lambda\ \rho \end{Bmatrix}\begin{Bmatrix} \tau \\ \sigma\ \tau \end{Bmatrix} \right) \\ & + g^{\nu\lambda}g^{\rho\sigma}\left(\begin{Bmatrix} \mu \\ \lambda\ \tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \rho\ \sigma \end{Bmatrix} + \begin{Bmatrix} \mu \\ \rho\ \sigma \end{Bmatrix}\begin{Bmatrix} \tau \\ \lambda\ \tau \end{Bmatrix} - \begin{Bmatrix} \mu \\ \sigma\ \tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \lambda\ \rho \end{Bmatrix} - \begin{Bmatrix} \mu \\ \lambda\ \rho \end{Bmatrix}\begin{Bmatrix} \tau \\ \sigma\ \tau \end{Bmatrix} \right) \\ & + g^{\lambda\rho}g^{\sigma\tau}\left(\begin{Bmatrix} \mu \\ \lambda\ \sigma \end{Bmatrix}\begin{Bmatrix} \nu \\ \rho\ \tau \end{Bmatrix} - \begin{Bmatrix} \mu \\ \lambda\ \rho \end{Bmatrix}\begin{Bmatrix} \nu \\ \sigma\ \tau \end{Bmatrix} \right) . \end{aligned} \tag{A.16}$$

To lowest order, this takes the form

$$\begin{aligned} 2\kappa(-g)t_{\mathrm{LL}}^{\mu\nu} = \eta^{\mu\nu} & \left(\frac{1}{2}\partial^{\lambda}\bar{h}^{\rho\sigma}\partial_{\rho}\bar{h}_{\sigma\lambda} + \frac{1}{8}\partial^{\rho}\bar{h}\partial_{\rho}\bar{h} - \frac{1}{4}\partial^{\rho}\bar{h}^{\sigma\lambda}\partial_{\rho}\bar{h}_{\sigma\lambda} \right) + \frac{1}{2}\partial^{\mu}\bar{h}^{\rho\sigma}\partial^{\nu}\bar{h}_{\rho\sigma} \\ & - \frac{1}{4}\partial^{\mu}\bar{h}\partial^{\nu}\bar{h} - \partial^{\mu}\bar{h}_{\rho\sigma}\partial^{\rho}\bar{h}^{\nu\sigma} - \partial^{\rho}\bar{h}^{\mu\sigma}\partial^{\nu}\bar{h}_{\rho\sigma} + \partial_{\rho}\bar{h}^{\mu\sigma}\partial^{\rho}\bar{h}^{\nu}{}_{\sigma} \ . \end{aligned} \tag{A.17}$$

B Angular Momentum Loss Derived from Energy-Momentum Loss

Following the argument given in problem 3 in Section 110 of Ref. [14], we derive the time average of the orbital angular momentum loss for Newtonian point masses from the time average of the dynamical energy loss given by Eq. (5.5). The result supports the discussion of the "spin" angular momentum loss given in §5.1.2.

We represent the time average of the energy loss of the system as the work of the "frictional forces" $\boldsymbol{\varrho}$ acting on the Newtonian point masses:

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{a=1}^{N} \left\langle \boldsymbol{\varrho}_a \cdot \dot{\boldsymbol{\xi}}_a \right\rangle . \tag{B.1}$$

The time average of the loss of angular momentum, $\boldsymbol{l} \stackrel{\text{def}}{=} \sum_{a=1}^{N} \boldsymbol{\xi}_a \times m_a \dot{\boldsymbol{\xi}}_a$, is given by

$$\left\langle \frac{dl_\alpha}{dt} \right\rangle = \sum_{a=1}^{N} \langle (\boldsymbol{\xi}_a \times \boldsymbol{\varrho}_a)_\alpha \rangle = \sum_{a=1}^{N} \epsilon_{\alpha\beta\gamma} \left\langle \xi_a^\beta \varrho_a^\gamma \right\rangle , \tag{B.2}$$

where the symbol $\epsilon_{\alpha\beta\gamma}$ denotes the three-dimensional antisymmetric tensor with $\epsilon_{123} = \epsilon^{123} = 1$. Note that a Newtonian point mass in ENGR is subject to Newton's equation of motion. To determine $\boldsymbol{\varrho}_a$, we write Eq. (5.5) as

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{G}{5c^5} \left\langle \dddot{D}_{\alpha\beta} \dddot{D}_{\alpha\beta} - \frac{1}{3} \dddot{D}_{\alpha\alpha} \dddot{D}_{\beta\beta} \right\rangle = -\frac{G}{5c^5} \left\langle \dot{D}_{\alpha\beta} D^{(\mathrm{v})}_{\alpha\beta} - \frac{1}{3} \dot{D}_{\alpha\alpha} D^{(\mathrm{v})}_{\beta\beta} \right\rangle , \tag{B.3}$$

where we have used the fact that the average values of the total time derivatives vanish. Here, $D^{(\mathrm{v})}_{\alpha\beta}$ represents the fifth-order derivative with respect to t. Substituting $\dot{D}_{\alpha\beta} = \sum_{a=1}^{N} m_a (\dot{\xi}_a^\alpha \xi_a^\beta + \xi_a^\alpha \dot{\xi}_a^\beta)$ into Eq. (B.3) and comparing with Eq. (B.1), we find

$$\varrho_a^\alpha = -\frac{2G}{5c^5} \left(D^{(\mathrm{v})}_{\alpha\beta} - \frac{1}{3} \delta^{\alpha\beta} D^{(\mathrm{v})}_{\gamma\gamma} \right) m_a \xi_a^\beta . \tag{B.4}$$

Substitution of Eq. (B.4) into Eq. (B.2) gives the result

$$\left\langle \frac{dl_\alpha}{dt} \right\rangle = -\frac{2G}{5c^5} \epsilon_{\alpha\beta\gamma} \left\langle \left(D^{(\mathrm{v})}_{\gamma\delta} - \frac{1}{3} \delta^{\gamma\delta} D^{(\mathrm{v})}_{\epsilon\epsilon} \right) D_{\beta\delta} \right\rangle = -\frac{2G}{5c^5} \epsilon_{\alpha\beta\gamma} \left\langle \ddot{D}_{\beta\delta} \dddot{D}_{\delta\gamma} \right\rangle . \tag{B.5}$$

This is equivalent to Eq. (5.16), with Eq. (5.13) satisfying the condition (5.17).

References

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *GRAVITATION* (Freeman, New York, 1973).

[2] R. Geroch and J. Winicour, J. Math. Phys. **22** (1981), 803.

[3] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, Phys. Rep. **258** (1995), 1.

[4] T. Kawai, Gen. Relat. Gravit. **18** (1986), 995 [Errata; **19** (1987), 1285].

[5] T. Kawai and N. Toma, Prog. Theor. Phys. **85** (1991), 901.

[6] T. Kawai, Phys. Rev. D **62** (2000), 104014.

[7] T. Kawai, Prog. Theor. Phys. **79** (1988), 920.

[8] T. Kawai and H. Saitoh, Prog. Theor. Phys. **81** (1989), 280.

[9] T. Kawai and H. Saitoh, Prog. Theor. Phys. **81** (1989), 1119.

[10] M. Schweizer and N. Straumann, Phys. Lett. A **71** (1979), 493.

[11] M. Schweizer, N. Straumann and A. Wipf, Gen. Relat. Gravit. **12** (1980), 951.

[12] T. Kawai, Prog. Theor. Phys. **82** (1989), 850.

[13] K. Hayashi and T. Shirafuji, Phys. Rev. D**19** (1979), 3524.

[14] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975).

[15] S. Weinberg, *GRAVITATION AND COSMOLOGY: PRINCIPLES AND APPLICATIONS OF THE GENERAL THEORY OF RELATIVITY* (John Wiley & Sons, New York, 1972).

[16] R. A. Hulse and J. H. Taylor, Astrophys. J. **195** (1975), L51.

[17] T. Shirafuji and G. G. L. Nashed, Prog. Theor. Phys. **98** (1997), 1355.

[18] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, Phys. Rev. Lett. **84** (2000), 4533.

[19] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toríbio and K. H. Castello Branco, gr-qc/0008073.

[20] J. W. Maluf and J. F. da Rocha-Neto, Phys. Rev. D **64** (2001), 084014.

[21] Y. Itin, gr-qc/0103017.

[22] M. Blagojevic and M. Vasilic, Class. Quantum Grav. **17** (2000), 3785.

[23] M. Blagojevic and M. Vasilic, Phys. Rev. D **64** (2001), 044010.